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SOLUTION OF THE INCOMPRESSIBLE MASS AND MOMENTUM EQUATIONS BY APPLICATION OF A COUPLED EQUATION LINE SOLVER

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SUMMARY

This paper describes an iterative technique for solving the coupled algebraic equations for mass and momentum conservation for an incompressible fluid flow. The technique is based on the simultaneous solution for pressure and velocity along lines. In a manner similar to ADI methods for a single variable, the solution domain is entirely swept line-by-line in each co-ordinate direction successively until a converged solution is obtained. The tight coupling between the equations that is guaranteed by the method results in an economical solution of the equation set.

KEY WORDS Incompressible Flow Coupled Line Gauss Seidel Two Dimensional

INTRODUCTION

Most numerical procedures for predicting fluid flows subdivide the solution domain into discrete volumes and base predictions on the satisfaction of the governing conservation constraints over these volumes. For incompressible fluid flows, solution difficulties often arise because pressure does not appear in its constraint equation, the continuity equation. The pressure-velocity (p–V) coupling problem has received considerable previous attention (see Reference 1 for a review). The present paper presents a simple, robust, and efficient method for dealing with the p–V coupling.

Background

The conservation balances over the discrete volumes are expressed as a set of coupled non-linear algebraic equations for each dependent variable.[2] A solution is achieved by solving, to some specified tolerance, the linear (fixed coefficient) set of equations, updating the coefficients, and repeating this cycle until the non-linear equations are satisfied. The equations that require solution for a given set of coefficients are referred to here as the 'linear set'. This section reviews solution methods for incompressible flows where the equations exhibit the p–V coupling problem.

Various direct solvers could be used to solve the full linear set. Except for small problems, such an approach is unattractive owing to excessive demands on computer storage and execution time.

Segregated solution procedures (e.g. SIMPLER[2] or SIMPLEC[3]) are most widely used to solve the linear set. In these, the subset of momentum equations for each velocity component is solved separately, usually by some iterative method, using an estimate of the pressure. The pressure and velocities are then corrected by solving another equation set that enforces mass conservation. This

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cycle may be repeated to account for the coupling between the segregated equation sets, but usually one cycle is most cost-effective.

In another procedure, iterative solvers are applied, whereby the values of all variables are simultaneously improved. Although somewhat more complex, such methods implicitly account for inter-variable couplings. Blottner[4] used a 'block ADI' procedure to solve the coupled boundary layer equations. Van Doormaal *et al.*[5] and Rubin and Khosla[6] solved the full two-dimensional (2D) incompressible flow equations expressed in terms of stream function and vorticity.

Only very recently have such solvers been applied to solve the 2D equations in the primitive variables u, v and p. Van Doormaal and Raithby[7] solved the momentum and continuity equations simultaneously along lines using the 'raw' form of continuity (zero velocity divergence). Zedan and Schneider[8–11] obtained an equation for pressure by substituting the algebraic momentum equations into the continuity equation. This pressure equation, together with the momentum equations, was solved by iteration using a number of coupled equation solvers. In the first of these two approaches, mass conservation is always exactly satisfied while the iteration improves the satisfaction of the momentum equations. In the second approach, the satisfaction of both mass and momentum conservation is improved by iteration. The relative advantages of the two approaches are not yet clear.

Objectives and attributes of a new coupled equation line solver (CELS)

The present study was directed at reformulating the method proposed by Van Doormaal and Raithby.[7] Deficiencies of this method included the complexity of the derivation and coding of the equations, and the relatively large computer overhead associated with the storage of coefficients. Another major detraction was the inability to prescribe one or more local pressures within the solution domain. This complicated both the prescription of a reference pressure and the prediction of flows driven by a pressure difference. The present reformulation resulted in a coupled equation line solver (CELS) for the primitive-variable equations of mass and momentum, with the following attributes:

1. The iterative procedure always enforces mass conservation exactly. The iterative improvement of variables is based on the direct solution of the mass and momentum equations along lines.
2. The CELS method is simple to derive and is easier to code than many of the commonly used segregated solution procedures.
3. The solver permits a direct specification of pressure wherever desired.
4. The solver is extremely robust for the problems investigated to date.

This paper limits attention to the solution of the two-dimensional incompressible flow equations. The method is not restricted to just the pressure–velocity coupling and has been extended to other couplings such as those that occur in natural convection problems between momentum and energy conservation. Extensions to three dimensions are also possible.

EQUATIONS OF MOTION

The conservation equations for mass and momentum for a steady, two-dimensional, incompressible flow can be expressed in the following differential form:[2]

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

$$\frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2}$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right). \tag{3}$$

In these equations u and v are the velocity components in the x and y directions, p is pressure, and μ is either the laminar or effective (laminar plus turbulent) viscosity. In a manner consistent with the 'incompressible' flow formulation, the density ρ is taken as constant.

To derive the algebraic analogue of equations (1)–(3), a mesh is overlaid on the solution domain and any of a number of discretization methods applied. In this paper the staggered mesh of Harlow and Welch[12] is used, so that velocities and pressures are stored at the locations indicated in Figure 1. The algebraic equations for continuity for the volume having p_{ij} at its centre, and the momentum equations for u_{ij} and v_{ij} can be written in the following form:

$$0 = A_{\rm E}^{\rm c} u_{ij} + A_{\rm W}^{\rm c} u_{i-1j} + A_{\rm N}^{\rm c} v_{ij} + A_{\rm S}^{\rm c} v_{ij-1}, \tag{4}$$

$$A_{\rm P}^{\rm u} u_{ij} = A_{\rm E}^{\rm u} u_{i+1j} + A_{\rm W}^{\rm u} u_{i-1j} + A_{\rm N}^{\rm u} u_{ij+1} + A_{\rm S}^{\rm u} u_{ij-1} - C^{\rm u}(p_{i+1j} - p_{ij}) + B^{\rm u}, \tag{5a}$$

$$A_{\rm P}^{\rm v} v_{ij} = A_{\rm E}^{\rm v} v_{i+1j} + A_{\rm W}^{\rm v} v_{i-1j} + A_{\rm N}^{\rm v} v_{ij+1} + A_{\rm S}^{\rm v} v_{ij-1} - C^{\rm v}(p_{ij+1} - p_{ij}) + B^{\rm v}, \tag{6a}$$

where

$$A_{\rm P}^{\rm u} = \left(1 + \frac{1}{E}\right)\sum A_{\rm nb}^{\rm u}, \qquad B^{\rm u} = \frac{1}{E}\sum A_{\rm nb}^{\rm u} u_{ij}^{\rm o}, \tag{5b}$$

$$A_{\rm P}^{\rm v} = \left(1 + \frac{1}{E}\right)\sum A_{\rm nb}^{\rm v}, \qquad B^{\rm v} = \frac{1}{E}\sum A_{\rm nb}^{\rm v} v_{ij}^{\rm o}. \tag{6b}$$

There is one continuity equation for each interior control volume in Figure 1 and one momentum

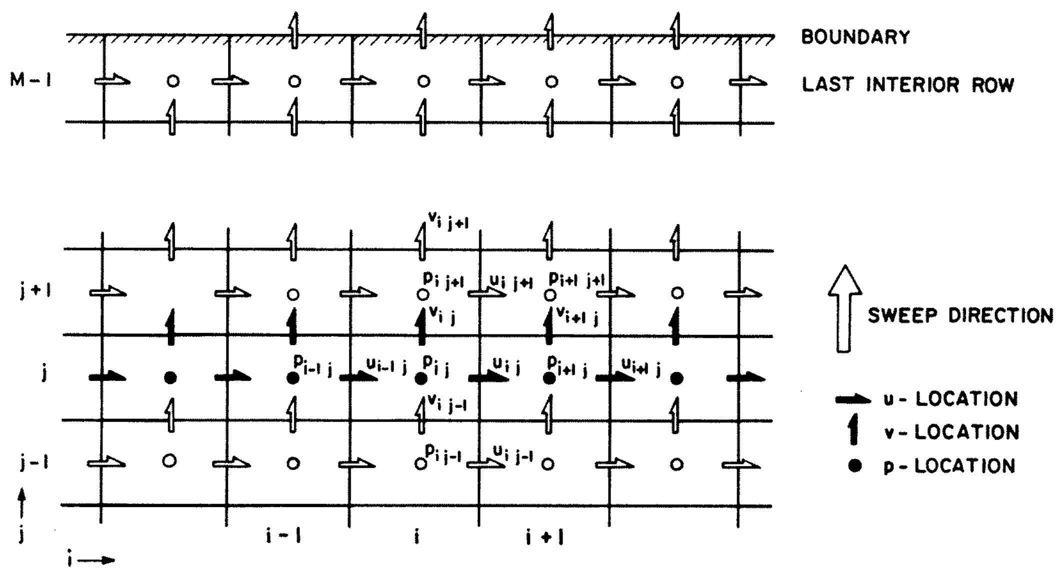

Figure 1. Grid layout showing storage location for variables u, v and p. The variables obtained in a simultaneous solution along the j-line are indicated by solid symbols

equation for each interior velocity shown. The application of boundary conditions provides additional algebraic equations for the velocity component that lies normal to and on the boundary, and for the component that lies tangent to and outside the boundary.

Equations (1)–(3) are non-linear and this is reflected by a dependence of the coefficients in equations (4)–(6) on the dependent variables. As described earlier, the linear equation set is solved using these fixed coefficients to obtain still better estimates of the dependent variables, the coefficients are then updated and the cycle repeated until convergence is achieved. To accelerate convergence or prevent divergence of the coefficient update loop, the E factor[1,3] is used to introduce relaxation into equations (5) and (6). The physical interpretation and advantages of the E-factor formulation are discussed elsewhere.[3] It is sufficient to point out here that, to justify the use of implicit (as opposed to fully explicit) methods, a solution algorithm should perform well for values of E in excess of 2.

The coefficients of the algebraic equations for the discrete velocities and pressures (equations (4)–(6)) were derived in this study using the finite volume method described by Patankar.[2] Two minor variations involved the use of the upstream-weighted differencing scheme of Raithby and Torrance,[13] and the E-factor formulation as shown above.

PROPOSED SOLUTION METHOD

Derivation of the coupled equation line solver (CELS)

Line Gauss Seidel (LGS) methods (or relaxation forms of ADI methods) are frequently applied to solve the equations for a single variable. The term LGS is intended here to include any method which solves an equation directly along a line, and iteratively improves the solution by successively sweeping the domain (solving along each line in a given sweep) in alternating directions. The present CELS method falls within this expanded definition of LGS methods in that three coupled equations (4)–(6) are solved directly along each line.

Attention is now turned to the derivation of the method for solving these coupled equations along a line.

A direct solution along the line of constant j in Figure 1 results in new values of all the variables indicated by solid symbols in this Figure. The pressures and velocities off the line, designated by open symbols in Figure 1, are held fixed at their most recent estimates. Along a line of constant j, equations (4)–(6) can be written as

$$A_{\mathrm{P}}^{\mathrm{u}}u_i = A_{\mathrm{E}}^{\mathrm{u}}u_{i+1} + A_{\mathrm{W}}^{\mathrm{u}}u_{i-1} + C^{\mathrm{u}}(p_i - p_{i+1}) + b^{\mathrm{u}}, \tag{7a}$$

$$A_{\mathrm{P}}^{\mathrm{v}}v_i = A_{\mathrm{E}}^{\mathrm{v}}v_{i+1} + A_{\mathrm{W}}^{\mathrm{v}}v_{i-1} + C^{\mathrm{v}}p_i + b^{\mathrm{v}}, \tag{8a}$$

$$0 = A_{\mathrm{E}}^{\mathrm{c}}u_i + A_{\mathrm{W}}^{\mathrm{c}}u_{i-1} + A_{\mathrm{N}}^{\mathrm{c}}v_i + b^{\mathrm{c}}, \tag{9a}$$

where the new source terms are

$$b^{\mathrm{u}} = B^{\mathrm{u}} + A_{\mathrm{N}}^{\mathrm{u}}u_{ij+1}^{*} + A_{\mathrm{S}}^{\mathrm{u}}u_{ij-1}^{*}, \tag{7b}$$

$$b^{\mathrm{v}} = B^{\mathrm{v}} + A_{\mathrm{N}}^{\mathrm{v}}v_{ij+1}^{*} + A_{\mathrm{S}}^{\mathrm{v}}v_{ij-1}^{*} - C^{\mathrm{v}}p_{ij+1}^{*}, \tag{8b}$$

$$b^{\mathrm{c}} = A_{\mathrm{S}}^{\mathrm{c}}v_{ij-1}^{*}. \tag{9b}$$

Subscripts of i and j are implied on all coefficients and variables unless otherwise explicitly indicated. The upper case coefficients are the original coefficients in equations (4)–(6). The lower case coefficients apply along a line of constant j. Variables denoted with superscript * indicate their most recent values. These conventions are used for the remainder of the derivation.

Step 1. The first step in the derivation is to use the three continuity equations, (9), for the i, $i+1$ and $i-1$ volumes to eliminate, respectively, v_i, v_{i+1} and v_{i-1} from equation (8a). The resulting equation, rearranged to isolate p_i, is

$$a_{\rm P}^{\rm P} p_i = a_{\rm EE}^{\rm P} u_{i+1} + a_{\rm E}^{\rm P} u_i + a_{\rm W}^{\rm P} u_{i-1} + a_{\rm WW}^{\rm P} u_{i-2} + b^{\rm P}, \tag{10a}$$

or, more compactly,

$$a_{\rm P}^{\rm P} p_i = \sum_{i-2}^{i+1} a_{\rm nb}^{\rm P} u_{\rm nb} + b^{\rm P}, \tag{10b}$$

where the coefficients are

$$a_{\rm P}^{\rm P} = c^{\rm v}, \tag{10c}$$

$$a_{\rm EE}^{\rm P} = A_{\rm E}^{\rm v} \frac{A_{{\rm E}i+1}^{\rm c}}{A_{{\rm N}i+1}^{\rm c}}, \tag{10d}$$

$$a_{\rm E}^{\rm P} = \left[A_{\rm E}^{\rm v} \frac{A_{{\rm W}i+1}^{\rm c}}{A_{{\rm N}i+1}^{\rm c}} - A_{\rm P}^{\rm v} \frac{A_{\rm E}^{\rm c}}{A_{\rm N}^{\rm c}} \right], \tag{10e}$$

$$a_{\rm W}^{\rm P} = \left[A_{\rm W}^{\rm v} \frac{A_{{\rm E}i-1}^{\rm c}}{A_{{\rm N}i-1}^{\rm c}} - A_{\rm P}^{\rm v} \frac{A_{\rm W}^{\rm c}}{A_{\rm N}^{\rm c}} \right], \tag{10f}$$

$$a_{\rm WW}^{\rm P} = A_{\rm W}^{\rm v} \frac{A_{{\rm W}i-1}^{\rm c}}{A_{{\rm N}i-1}^{\rm c}}, \tag{10g}$$

$$b^{\rm P} = \left[A_{\rm E}^{\rm v} \frac{b_{i+1}^{\rm c}}{A_{{\rm N}i+1}^{\rm c}} + A_{\rm W}^{\rm v} \frac{b_{i-1}^{\rm c}}{A_{{\rm N}i-1}^{\rm c}} - A_{\rm P}^{\rm v} \frac{b^{\rm c}}{A_{\rm N}^{\rm c}} - b^{\rm v} \right]. \tag{10h}$$

Step 2. The second step is to use the derived pressure equation, (10a), for the i and $i+1$ volumes to eliminate, respectively, p_i and p_{i+1} from equation (7a). The following penta-diagonal equation for u_i results:

$$a_{\rm P}^{\rm u} u_i = a_{\rm EE}^{\rm u} u_{i+2} + a_{\rm E}^{\rm u} u_{i+1} + a_{\rm W}^{\rm u} u_{i-1} + a_{\rm WW}^{\rm u} u_{i-2} + b^{*\rm u}, \tag{11a}$$

or, more compactly,

$$a_{\rm P}^{\rm u} u_i = \sum_{i-2}^{i+2} a_{\rm nb}^{\rm u} u_{\rm nb} + b^{*\rm u}, \tag{11b}$$

where the coefficients are

$$a_{\rm P}^{\rm u} = A_{\rm P}^{\rm u} + C^{\rm u} \left[\frac{a_{{\rm W}i+1}^{\rm P}}{a_{{\rm P}i+1}^{\rm P}} - \frac{a_{\rm E}^{\rm P}}{a_{\rm P}^{\rm P}} \right], \tag{11c}$$

$$a_{\rm EE}^{\rm u} = -C^{\rm u} \frac{a_{{\rm EE}i+1}^{\rm P}}{a_{{\rm P}i+1}^{\rm P}}, \tag{11d}$$

$$a_{\rm E}^{\rm u} = A_{\rm E}^{\rm u} + C^{\rm u} \left[\frac{a_{\rm EE}^{\rm P}}{a_{\rm P}^{\rm P}} - \frac{a_{{\rm E}i+1}^{\rm P}}{a_{{\rm P}i+1}^{\rm P}} \right], \tag{11e}$$

$$a_{\rm W}^{\rm u} = A_{\rm W}^{\rm u} + C^{\rm u} \left[\frac{a_{\rm W}^{\rm P}}{a_{\rm P}^{\rm P}} - \frac{a_{{\rm WW}i+1}^{\rm P}}{a_{{\rm P}i+1}^{\rm P}} \right], \tag{11f}$$

$$a_{\rm WW}^{\rm u} = C^{\rm u} \frac{a_{\rm WW}^{\rm P}}{a_{\rm P}^{\rm P}}, \tag{11g}$$

$$b^{*u} = b^u + C^u\left[\frac{b^P}{a_P^P} - \frac{b_{i+1}^P}{a_{Pi+1}^P}\right]. \tag{11h}$$

A penta-diagonal solver can be used to efficiently solve equation (11a) (see Appendix).

With each u-velocity along the line now known, a direct substitution into equation (9a) yields values of v_i, and into equation (10a) yields values of p_i. Computational effort is required to calculate the coefficients of equations (9)–(11), which are stored for future use, and to solve in the proposed manner. This effort only marginally exceeds that required for three uncoupled tri-diagonal solutions of u, v and p as in a segregated solution procedure. The additional storage requirements for the proposed method are approximately equivalent to the requirements for storing the coefficients of equations (4)–(6). This storage is several times that required for segregated methods but is significantly less than other simultaneous solution methods.

Pressure can be specified at any point by setting all the coefficients on velocity in equation (10a) to zero, and by replacing b^p by $a_P^p p_{spec}$, where p_{spec} is the specified pressure. This replaces a continuity equation by the pressure specification as discussed in Reference 3.

Solution method

To improve the solution for u, v and p over the entire calculation domain, the grid is swept line-by-line in one direction (e.g. the y-sweep direction in Figure 1), followed by a similar sweep in the other direction. Improved estimates of the dependent variables from one line sweep are used as the best available off-line values in the next line solution. The sweeps are repeated, for a given set of coefficients in equations (4)–(6), until some convergence criterion is met. The residual reduction criterion described in Reference 3 was used in the present study.

Special treatment on the last line. In the implementation of the present procedure, special attention is required on the last line ($j = M - 1$ in Figure 1) because the equations for the v velocities on the boundary are the boundary conditions and not, as for other lines, equation (8a). The v_{iM-1} velocities are therefore directly found from the boundary conditions. Mass conservation applied to each volume directly yields the u_{iM-1} velocities. Equation (6) is then applied for $j = M - 2$ to obtain the p_{iM-1} pressures along the last line.

Block correction of pressure. After a solution along the $j - 1$ line in Figure 1, the v_{ij-1} velocities are held fixed when CELS is applied to the j line. This decouples the pressures between these lines, leaving the level of p on the j line to be determined by pressures on the $j + 1$ line. This decoupling slows the convergence of the solver, and can be largely avoided by adjusting the pressure levels of each line after each sweep is completed. For the j-sweep in Figure 1, this requires adding δp_j to the pressure in each line; the constraint that determines δp_j is that the v-momentum equations must be satisfied on the average along the j line. If the pressures p_{ij} along the $j + 1$ line have already been corrected, δp_j is obtained from

$$\delta p_j = \left[\sum_{\text{all } i}\left(A_p^v v_{ij} - \sum_{\text{nb}} A_{\text{nb}}^v v_{\text{nb}} + C^v(p_{ij+1} - p_{ij}) - b^v\right)\right] \Big/ \sum_{\text{all } i} C^v. \tag{12}$$

Each pressure on the j line is then adjusted using

$$p_{ij} = p_{ij}^* + \delta p_j, \quad \text{for all} \quad i. \tag{13}$$

This procedure is repeated for each j beginning at $j = M - 2$ and ending on the first j line interior to the solution domain.

Relaxation within the solver. The solution along each line would yield the exact solution to the problem if the variables ahead of the line were correct. In the above derivation of the coupled solver, it was assumed that the most recent (approximate) values were used both behind and ahead of the line.

When a small E is used in the generation of the coefficients of the linear set, significant under-relaxation is present such that the effective radius of influence of pressure on momentum is very local, as discussed by Zedan and Schneider.[8] In this instance the approximations introduced in forming the coupled line equations (i.e. fixing the off-line variables at their best estimates) is physically consistent and thus CELS converges monotonically and rapidly.

When a large E, ($E \gtrsim 20$), is used in the generation of the coefficients, the radius of influence of pressure increases, and iterative procedures generally become less effective. In this instance, CELS becomes slow to converge, as the iterative solution is now very sensitive to the estimates of off-line variables. Experience has shown that the introduction of a simple relaxation into CELS considerably enhances convergence at high E. The relaxation is introduced by modifying the linearized momentum equations as follows (for a line of constant j):

$$A_{\mathrm{P}}^{\mathrm{u}}\left(1+\frac{1}{e}\right)u_i = A_{\mathrm{E}}^{\mathrm{u}}u_{i+1} + A_{\mathrm{W}}^{\mathrm{u}}u_{i-1} + C^{\mathrm{u}}(p_i - p_{i+1}) + b^{\mathrm{u}} + \frac{A_{\mathrm{P}}^{\mathrm{u}}}{e}u_i^*, \tag{14a}$$

$$A_{\mathrm{P}}^{\mathrm{v}}\left(1+\frac{1}{e}\right)v_i = A_{\mathrm{E}}^{\mathrm{v}}v_{i+1} + A_{\mathrm{W}}^{\mathrm{v}}v_{i-1} + C^{\mathrm{v}}p_i + b^{\mathrm{v}} + \frac{A_{\mathrm{P}}^{\mathrm{v}}}{e}v_i^*. \tag{14b}$$

The values of u^* and v^* are the values of u and v calculated in the last sweep of the solution domain, and e is the linear equation *solver* relaxation parameter that is analogous to E used in the coefficient update loop. When CELS converges, $u_i = u_i^*$ and $v_i = v_i^*$, and the relaxation-related terms disappear. For the tests to date, it has been observed that CELS performs best with *no* solver relaxation ($e = \infty$) for $E \lesssim 20$, and with $e \approx 5$ for $E \gtrsim 20$, but the solver is insensitive to the precise value of e used.

TEST OF SOLUTION METHODS

Test problems

For the purpose of demonstrating the applicability of the proposed CELS method and to evaluate its performance relative to existing segregated methods, two fluid flow problems were solved. An internal flow problem, involving laminar flow in a rectangular tank with a 4:1 aspect ratio, was solved using a 22×22 grid (20×20 interior mass control volumes). The geometry, boundary conditions and velocity vectors for this problem are depicted in Figure 2(A). As an external flow problem, the laminar flow of air over a rearward facing step was predicted using a 27×17 grid (25×15 interior mass control volumes); the geometry and velocities for this problem are shown in Figure 2(B).

Three solution methods were tested: the SIMPLE and SIMPLER methods of Patankar[2] and CELS. The performance of SIMPLE was enhanced by using the SIMPLEC[3] (or consistent 'time step'[1]) formulation.

For each problem the 'exact' solution to the discrete equations, for the chosen grid and discrete method,[2] was first determined. To within machine accuracy, this solution is independent of the solver used. The velocity and pressure fields were then initialized to zero and the computational effort required for each solution method to achieve an accuracy of $\pm 0{\cdot}5$ per cent was determined for both problems. Accuracy was defined as the maximum error in the pressure field normalized by the range of pressure in the exact solution.

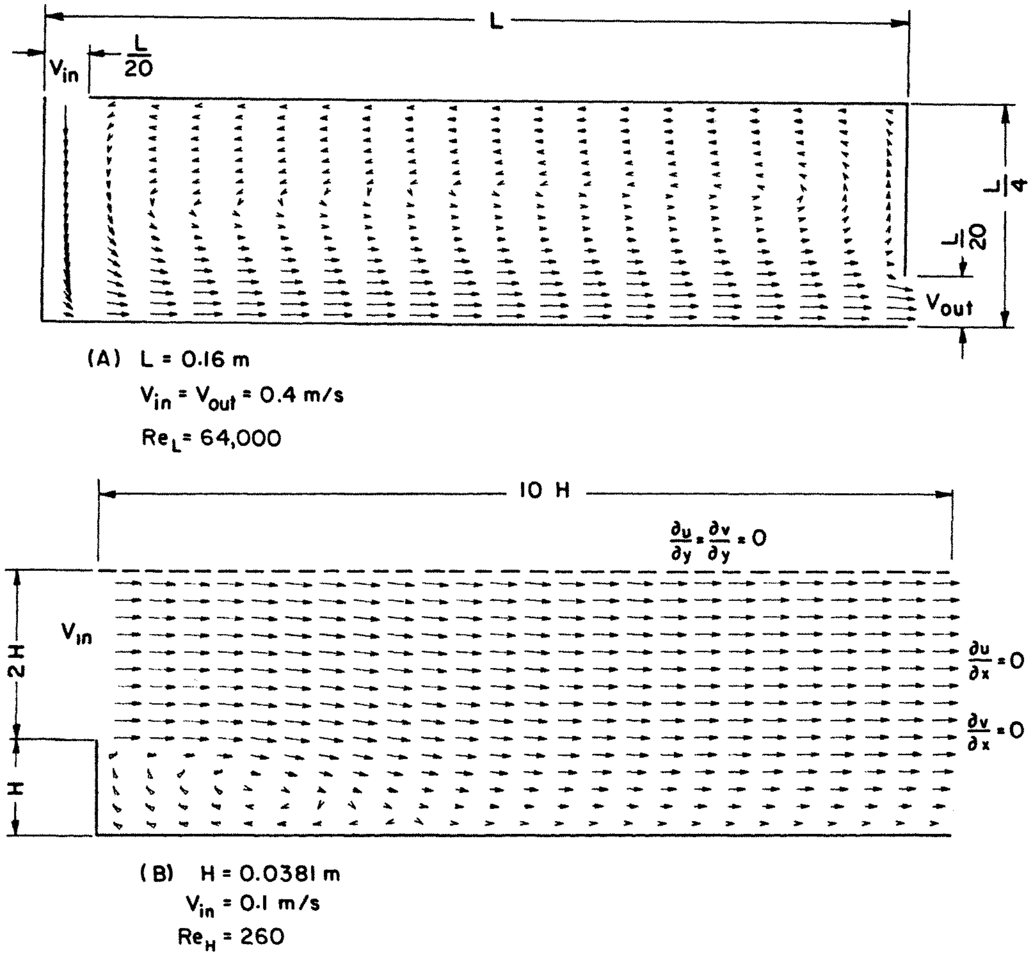

Figure 2. Velocity vectors for the (A) confined water flow and (B) external air flow test problem. Velocities are zero on all solid boundaries. The prescribed velocities at the inlet and outlet boundaries are uniform

The computational effort required for each of the methods to satisfy the above requirement depends on a number of parameters including the relaxation factor E and the convergence criteria used to terminate the iterative solutions of equations for each fixed set of coefficients. The dependence of computational effort on E for each of the methods is reported below. The convergence criteria for each method were selected to yield minimum total computational effort at the optimal E value for the given method and problem. All calculations were carried out on an IBM 4341-Type 1 installation using single precision FORTRAN VS.

Results

The CPU times required to achieve the stated convergence criterion for SIMPLEC, SIMPLER and CELS, are shown in Figure 3(A) for the confined flow, and Figure 4(A) for the external flow. For large under-relaxation in the coefficients (small E), SIMPLEC outperforms SIMPLER, but the computational effort required by both methods increases dramatically for E values larger than 5. By comparison, the performance of CELS lies between SIMPLEC and SIMPLER at small E,

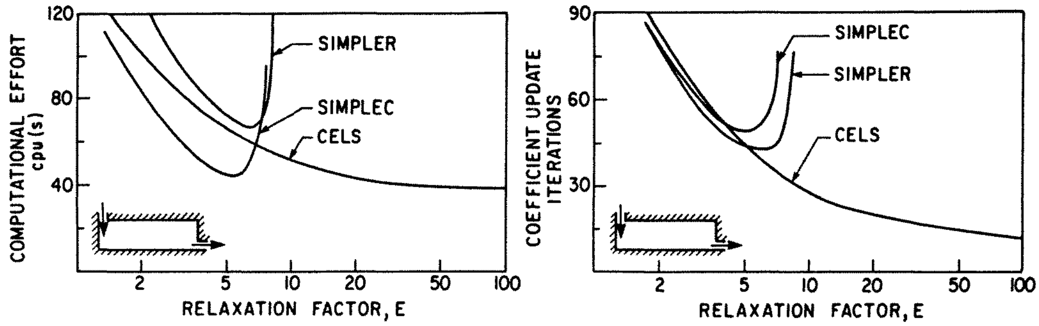

Figure 3. Computational effort (A) and number of coefficient update iterations (B) to achieve the prescribed accuracy in pressure vs. distorted time step parameter, E, for the confined flow

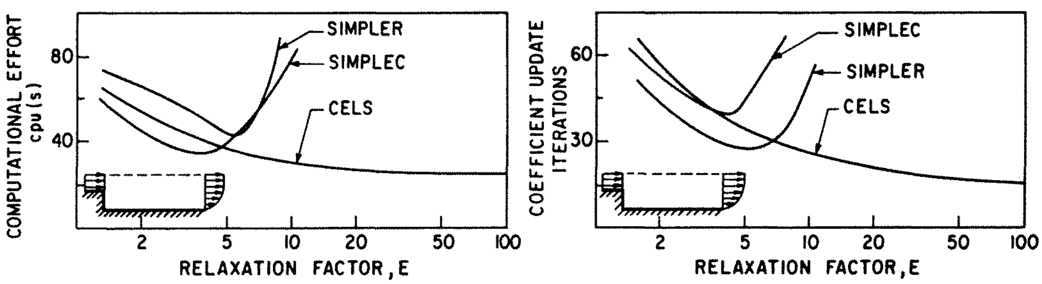

Figure 4. Computational effort (A) and number of coefficient update iterations (B) to achieve the prescribed accuracy in pressure vs. distorted time step parameter, E, for the external test flow problem

and decreases monotonically with E towards a constant value. The competitive performance relative to refined state-of-the-art solution methods at small E, the distinctly superior performance at large E, and the relative simplicity, combine to make CELS a highly attractive method.

SIMPLEC and SIMPLER have been implemented in the recommended manner whereby the velocity and pressure equation(s) are solved only once for each new set of coefficients. Insight into the relative performances of the three methods can also be gained by examining the number of coefficient updates required to reach the prescribed convergence criterion. These are shown in Figures 3(B) and 4(B) for the confined and external flows, respectively. Figure 3(B) shows that all three methods required roughly the same number of updates at small E so that the higher cost of SIMPLER results from the need to solve an extra pressure equation. The sharp increase in the number of coefficient updates for the segregated methods for $E \gtrsim 5$ reflects the breakdown in the pressure–velocity coupling approximation. CELS retains the proper coupling and this results in a steady decrease in the required number of coefficient updates with increasing E. For large E, the solution time (Figure 3(A)) does not continue to fall with the decrease in number of coefficient updates (Figure 3(B)) because of the corresponding increase in computational effort required to solve each linear set.

The results presented in Figures 3 and 4 were obtained using optimal values of the residual reduction criteria used to terminate iteration on the equations that are solved within the coefficient update loop, as previously described. It has been established, by running numerical experiments, that all three methods are about equally sensitive to departures of these criteria from their optimal value.

CLOSING REMARKS

A solution method has been described that is based on the exact solution of the momentum equations and continuity equation along a line. Tests of this method have demonstrated that such a solver is competitive, in terms of execution time and in terms of the simplicity of the derivation and the coding, when compared to state-of-the-art segregated solvers applied to complex incompressible flows. The method is also more robust in terms of its insensitivity to the major relaxation factor (relaxation at the coefficient update level) due to the improved handling of the pressure–velocity coupling.

Other couplings between the governing equations for incompressible flows can be of equal or greater importance than the pressure–velocity coupling. Examples include the increased inter-momentum equation coupling arising through the acceleration terms present in general orthogonal co-ordinates, or the temperature–velocity coupling in non-isothermal flows. Coupled line solvers based on the CELS procedure that include the above couplings are currently under investigation. Initial results indicate that the computational effort can be dramatically reduced from that of segregated methods.

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APPENDIX: PENTA-DIAGONAL MATRIX SOLUTION ALGORITHM (PDMA)

For completeness, the PDMA algorithm is given below. It is derived exactly as the tri-diagonal matrix algorithm (TDMA), except the PDMA derivation requires one further elimination.

In general, any penta-diagonal system of equations can be written as:

$$a_P\phi_i = a_{EE}\phi_{i+2} + a_E\phi_{i+1} + a_W\phi_{i-1} + a_{WW}\phi_{i-2} + b_P. \tag{15}$$

Equation (15) is solved by manipulating the above equation into the following upper diagonal (recursive) form:

$$\phi_i = A_i\phi_{i+2} + B_i\phi_{i+1} + C_i, \tag{16}$$

where

$$A_i = a_{EE}/F$$

$$B_i = (a_E + DA_{i-1})/F$$

$$C_i = (b_P + DC_{i-1} + a_{WW}C_{i-2})/F$$

$$D = a_W + a_{WW}B_{i-2}$$

$$F = a_P - a_{WW}A_{i-2} - DB_{i-1}$$

REFERENCES

1. G. D. Raithby and G. E. Schneider, 'Numerical solution of problems in incompressible fluid flow; treatment of the velocity–pressure coupling', *Num. Heat Transfer*, **2**, 417–440 (1979).
2. S. V. Patankar, *Numerical Heat Transfer and Fluid Flow*, Hemisphere, New York, 1980.

3. J. P. Van Doormaal and G. D. Raithby, 'Enhancements of the SIMPLE method for predicting incompressible fluid flows', *Num. Heat Transfer*, **7,** 147–163 (1984).
4. F. G. Blottner, 'Investigation of some finite-difference techniques for solving the boundary layer equations', *Num. Meth. Appl. Mech. Eng.*, **6,** 1–30 (1975).
5. J. P. Van Doormaal, G. D. Raithby and A. B. Strong, 'Prediction of natural convection in non-rectangular enclosures', *Num. Heat Transfer*, **4,** 21–38 (1981).
6. S. G. Rubin and P. K. Khosla, 'Navier–Stokes calculations with a coupled strongly implicit method-I', *Computers and Fluids*, **9,** 163–180 (1981).
7. J. P. Van Doormaal and G. D. Raithby, 'The simultaneous solution along lines of the continuity and momentum equations', *IMACS Conference*, Montreal, August 1982.
8. M. Zedan and G. E. Schneider, 'Investigation of the simultaneous variable solution for velocity and pressure in incompressible fluid flow problems', *AIAA 18th Thermophysics Conference*, Montreal, June 1983.
9. M. Zedan and G. E. Schneider, 'A strongly implicit simultaneous variable solution procedure for velocity and pressure in fluid flow problems', *AIAA 18th Thermophysics Conference*, Montreal, June 1983.
10. M. Zedan, 'Simultaneous variable solutions for velocity and pressure in fluid flow problems', *PhD Thesis*, University of Waterloo, Waterloo, Canada, 1984.
11. G. E. Schneider and M. Zedan, 'A coupled modified strongly implicit procedure for the numerical solution of coupled continuum problems', *AIAA 19th Thermophysics Conference,* Montreal, June 1984.
12. F. H. Harlow and J. E. Welch, 'Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface', *Physics of Fluids*, **8,** (12), 2182–2189 (1965).
13. G. D. Raithby and K. E. Torrance, 'Upstream-weighted differencing schemes and their application to elliptic problems involving fluid flow', *Computers and Fluids*, **2,** 191–206 (1974).